Lecture 4 Time-Independent Homogeneous BC

The boundary conditions in previous lecture are assumed to be homogeneous, where we are able to use the superposition principle in forming general solutions of the PDE. We now turn to the situation where the BC are not both homogeneous, but are independent of time variable t. The method of solution consists of the following steps:

- Step 1: Find a particular solution of the PDE and BC.
- Step 2: Find the solution of a related problem with homogeneous BC. Then, add this solution to that particular solution obtained in Step 1.

The procedure is illustrated in the following example.

$$\begin{aligned} \text{PDE:} \quad & u_t = \alpha^2 u_{xx} \quad 0 \le x \le L,\ 0 < t < \infty, & (1)\\ \text{BC:} \quad & u(0,t) = a \ \ u(L,t) = b, \quad 0 < t < \infty, & (2)\\ \text{IC:} \quad & u(x,0) = f(x), \quad 0 \le x \le L, & (3) \end{aligned}$$

where a and b are arbitrary constants and $f(x)$ is a given function.

Solution. Seek a particular solution $u_p(x,t)$ of the form $u_p(x,t) = cx + d$, where c and d are chosen so that the BC are satisfied:

$$\begin{aligned} & a = u_p(0,t) = c \cdot 0 + d = d,\\ & b = u_p(L,t) = cL + d = cL + a.\\ \implies \quad & d = a \quad \text{and} \quad c = (b-a)/L. \end{aligned}$$

Thus,

$$u_p(x,t) = (b-a)x/L + a$$

solves both the PDE with the BC's being satisfied.

Consider the related homogeneous problem (i.e., with homogeneous PDE and BC)

$$\begin{aligned} \text{PDE:} \quad & v_t = \alpha^2 v_{xx} \quad 0 \le x \le L,\ 0 < t < \infty,\\ \text{BC:} \quad & v(0,t) = 0, \ \ v(L,t) = 0, \quad 0 < t < \infty, \qquad (4)\\ \text{IC:} \quad & v(x,0) = f(x) - u_p(x,0), \quad 0 \le x \le L. \end{aligned}$$

If $f(x) - u_p(x,0)$ is of the form $\sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$, then its solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/L)^2\alpha^2 t} \sin(n\pi x/L).$$

Now, set $u(x,t) = u_p(x,t) + v(x,t)$. Then it is easy to verify that $u(x,t)$ solves (1). Indeed, $u(x,t)$ solves (1) by the superposition principle. Further, we have

$$\begin{aligned}
\text{BC:} \quad & u(0,t) = u_p(0,t) + v(0,t) = a + 0 = a \\
& u(L,t) = u_p(L,t) + v(L,t) = b + 0 = b \\
\text{IC:} \quad & u(x,0) = u_p(x,0) + v(x,0) = u_p(x,0) + f(x) - u_p(x,0) = f(x).
\end{aligned}$$

Remark 1. *(i) It is necessary to subtract $u_p(x,0)$ from $f(x)$ to form the initial condition for the related problem (4) so that the initial condition (3) is satisfied.*

(ii) Since any particular solution will do, for simplicity one should consider a particular solution of the form $cx + d$, and find the constants, using the BC. The reason is that the formula only applies to the BC of (2). For other BC, we obtain other particular solution. For example, If $u_x(0,t) = a,\ u(L,t) = b$ then $u_p(x,t) = a(x - L) + b$.

Example 2.

$$\textit{PDE:} \quad u_t = 2u_{xx} \quad 0 \le x \le 1,\ 0 < t < \infty, \tag{5}$$

$$\textit{BC:} \quad u_x(0,t) = 1 \ \ u(1,t) = -2, \quad 0 < t < \infty, \tag{6}$$

$$\textit{IC:} \quad u(x,0) = x + \cos^2(3\pi x/4) - 5/2. \tag{7}$$

Solution. Take $u_p(x,t) = cx + d$. The first BC $u_x(0,t) = 1$ yields $c = 1$, while $u_p(1,t) = 1 + d$ yields $d = -3$ by the second BC. Thus, $u_p(x,t) = x - 3$. The related homogeneous problem is

$$\begin{aligned}
& v_t = 2v_{xx} \quad 0 \le x \le 1,\ 0 < t < \infty, \\
& v_x(0,t) = 0 \ \ v(1,t) = 0, \quad 0 < t < \infty \\
& v(x,0) = [x + \cos^2(3\pi x/4) - 5/2] - (x - 3) \\
& \quad = \frac{1}{2} + \frac{1}{2}\cos(3\pi x/2) - 5/2 + 3 = 1 + \frac{1}{2}\cos(3\pi x/2).
\end{aligned}$$

It is easy to obtain the solution of the related homogeneous problem as

$$v(x,t) = e^{-9\pi^2 t/2}[1 + \frac{1}{2}\cos(3\pi x/2)].$$

Then

$$u(x,t) = x - 3 + e^{-9\pi^2 t/2}[1 + \frac{1}{2}\cos(3\pi x/2)].$$

From the above examples, we notice that the particular solution is time independent, or in steady-state.

Note: Any steady-state solution of the heat equation $u_t = \alpha^2 u_{xx}$ is of the form $cx + d$.

The solutions $u(x,t)$ are sums of a steady-state particular solution of the PDE and BC and the solution $v(x,t)$ of the related homogeneous problem which is transient in the sense that $v(x,t) \to 0$ as $t \to \infty$. Thus

$$\begin{array}{rcccc} u(x,t) & = & u_p(x,t) & + & v(x,t) \;\to\; u_p(x,t), \;\; \text{as } t \to \infty. \\ & & \text{(steady-state solution)} & & \text{(transient solution)} \end{array}$$

That is, the solution u approaches the steady-state solution as $t \to \infty$. However, for some types of BC there are no steady-state particular solutions, as illustrated in the following example.

Example 3. *Consider the problem*

$$\begin{aligned} \textit{PDE:} \quad & u_t = \alpha^2 u_{xx} \quad 0 \le x \le L,\; 0 < t < \infty, && (8)\\ \textit{BC:} \quad & u_x(0,t) = a \;\; u_x(L,t) = b, && (9)\\ \textit{IC:} \quad & u(x,0) = f(x), && (10) \end{aligned}$$

where a and b are constants, and $f(x)$ is a given function.

Solution. Let $u_p(x,t) = cx + d$. Then, using BC, we obtain $c = a$ and $c = b$, which is impossible unless $a = b$.

NOTE: Observe that the boundary conditions state that heat is being drained out of the end $x = 0$ at a rate $u_x(0,t) = a$ and heat is flowing into the end $x = L$ at a rate $u_x(L,t) = b$. If $b > a$, then the heat energy is being added to the rod at a constant rate. If $b < a$, the rod loses heat at a constant rate. Thus, we cannot expect a steady-state solution of the PDE and BC, unless $a = b$.

The simplest form for a particular solution, that reflects the fact that the heat energy is changing at a constant rate, is

$$u_p(x,t) = ct + h(x)$$

where c is a constant and $h(x)$ is a function of x. The constant c and the function $h(x)$ can be determined from the PDE and BC. Thus,

$$\begin{aligned} & c = (u_p)_t = \alpha^2 (u_p)_{xx} = \alpha^2 h''(x) \\ \Longrightarrow \quad & h''(x) = \frac{c}{\alpha^2} \\ \Longrightarrow \quad & h(x) = \frac{c}{2\alpha^2} x^2 + dx + e, \end{aligned}$$

for constants d and e. Using BC, we note that

$$a = (u_p)_x(0,t) = h'(0) = d \implies d = a.$$

$$b = (u_p)_x(L,t) = h'(L) = \frac{cL}{\alpha^2} + d \implies c = \frac{(b-a)\alpha^2}{L}.$$

Thus, a particular solution (taking $e = 0$, for simplicity) is obtained as:

$$u_p(x,t) = \frac{(b-a)}{L}\alpha^2 t + \frac{(b-a)}{2L}x^2 + ax = \frac{(b-a)}{L}[\alpha^2 t + \frac{1}{2}x^2] + ax. \tag{11}$$

The related homogeneous problem is

$$\begin{aligned}
&v_t = \alpha^2 v_{xx} \quad 0 \le x \le L,\ t \ge 0,\\
&v_x(0,t) = 0\ \ v_x(L,t) = 0, \quad 0 < t < \infty,\\
&v(x,0) = f(x) - u_p(x,0) = f(x) - [\frac{(b-a)}{2L}x^2 + ax].
\end{aligned}$$

If $f(x) - u_p(x,0)$ is of the form $\sum_{n=0}^{\infty} c_n \cos(n\pi x/L)$, we have the solution

$$\begin{aligned}
u(x,t) &= u_p(x,t) + v(x,t)\\
&= u_p(x,t) + \sum_{n=0}^{\infty} c_n e^{-(n\pi/L)^2\alpha^2 t}\cos(n\pi x/L),
\end{aligned}$$

where $u_p(x,t)$ is given by (11).

Practice Problems

1. Solve the following IBVP:

$$\begin{aligned}
&u_t = u_{xx}, \ \ 0 < x < L, \ \ t > 0,\\
&u(0,t) = a, \ u(L,t) = b, \quad t > 0,\\
&u(x,0) = a + bx, \ \ 0 \le x \le L.
\end{aligned}$$

2. Solve the following IBVP:

$$\begin{aligned}
&u_t = 4u_{xx}, \ \ 0 < x < \pi, \ \ t > 0,\\
&u(0,t) = 5, \ u(\pi,t) = 10, \quad t > 0,\\
&u(x,0) = \sin x - \sin 3x, \ \ 0 < x < \pi.
\end{aligned}$$