Lecture 3 Method of Separation of Variables

Separation of variables is one of the oldest technique for solving initial-boundary value problems (IBVP) and applies to problems, where

- PDE is linear and homogeneous (not necessarily constant coefficients) and
- BC are linear and homogeneous.

Basic Idea: To seek a solution of the form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ is some function of x and $T(t)$ in some function of t. The solutions are simple because any temperature $u(x,t)$ of this form will retain its basic "shape" for different values of time t. The separation of variables reduced the problem of solving the PDE to solving the two ODEs: One second order ODE involving the independent variable x and one first order ODE involving t. These ODEs are then solved using given initial and boundary conditions.

To illustrate this method, let us apply to a specific problem. Consider the following IBVP:

$$\text{PDE:} \qquad u_t = \alpha^2 u_{xx}, \quad 0 \le x \le L,\ 0 < t < \infty, \tag{1}$$

$$\text{BC:} \qquad u(0,t) = 0 \ \ u(L,t) = 0, \quad 0 < t < \infty, \tag{2}$$

$$\text{IC:} \qquad u(x,0) = f(x), \quad 0 \le x \le L. \tag{3}$$

Step 1:(Reducing to the ODEs) Assume that equation (1) has solutions of the form

$$\boxed{u(x,t) = X(x)T(t),}$$

where X is a function of x alone and T is a function of t alone. Note that

$$u_t = X(x)T'(t) \quad \text{and} \quad u_{xx} = X''(x)T(t).$$

Now, substituting these expression into $u_t = \alpha^2 u_{xx}$ and separating variables, we obtain

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\Rightarrow \qquad \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since a function of t can equal a function of x only when both functions are constant. Thus,

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = c$$

for some constant c. This leads to the following two ODEs:

$$T'(t) - \alpha^2 c T(t) = 0, \quad (4)$$

$$X''(x) - cX(x) = 0. \quad (5)$$

Thus, the problem of solving the PDE (1) is now reduced to solving the two ODEs.

Step 2:(Applying BCs)
Since the product solutions $u(x,t) = X(x)T(t)$ are to satisfy the BC (2), we have

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad X(L)T(t) = 0, \quad t > 0.$$

Thus, either $T(t) = 0$ for all $t > 0$, which implies that $u(x,t) = 0$, or $X(0) = X(L) = 0$. Ignoring the trivial solution $u(x,t) = 0$, we combine the boundary conditions $X(0) = X(L) = 0$ with the differential equation for X in (5) to obtain the BVP:

$$X''(x) - cX(x) = 0, \quad X(0) = X(L) = 0. \quad (6)$$

There are three cases: $c < 0$, $c > 0$, $c = 0$ which will be discussed below. It is convenient to set $c = -\lambda^2$ when $c < 0$ and $c = \lambda^2$ when $c > 0$, for some constant $\lambda > 0$.

Case 1. ($c = \lambda^2 > 0$ for some $\lambda > 0$). In this case, a general solution to the differential equation (5) is

$$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x},$$

where C_1 and C_2 are arbitrary constants. To determine C_1 and C_2, we use the BC $X(0) = 0$, $X(L) = 0$ to have

$$X(0) = C_1 + C_2 = 0, \quad (7)$$

$$X(L) = C_1 e^{\lambda L} + C_2 e^{-\lambda L} = 0. \quad (8)$$

From the first equation, it follows that $C_2 = -C_1$. The second equation leads to

$$\begin{aligned} & C_1(e^{\lambda L} - e^{-\lambda L}) = 0, \\ \Rightarrow \quad & C_1(e^{2\lambda L} - 1) = 0, \\ \Rightarrow \quad & C_1 = 0. \end{aligned}$$

since $(e^{2\lambda L}-1)>0$ as $\lambda>0$. Therefore, we have $C_1=0$ and hence $C_2=0$. Consequently $X(x)=0$ and this implies $u(x,t)=0$ i.e., there is no nontrivial solution to (5) for the case $c>0$.

Case 2. (when c=0)
The general solution solution to (5) is given by

$$X(x)=C_3+C_4x.$$

Applying BC yields $C_3=C_4=0$ and hence $X(x)=0$. Again, $u(x,t)=X(x)T(t)=0$. Thus, there is no nontrivial solution to (5) for $c=0$.

Case 3. (When $c=-\lambda^2<0$ for some $\lambda>0$)

The general solution to (5) is

$$X(x)=C_5\cos(\lambda x)+C_6\sin(\lambda x).$$

This time the BC $X(0)=0,\ \ X(L)=0$ gives the system

$$\begin{aligned} C_5 &= 0, \\ C_5\cos(\lambda L)+C_6\sin(\lambda L) &= 0. \end{aligned}$$

As $C_5=0$, the system reduces to solving $C_6\sin(\lambda L)=0$. Hence, either $\sin(\lambda L)=0$ or $C_6=0$. Now

$$\sin(\lambda L)=0 \quad\Longrightarrow\quad \lambda L=n\pi,\quad n=0,\pm1,\pm2,\ldots.$$

Therefore, (5) has a nontrivial solution ($C_6\neq0$) when

$$\lambda L=n\pi \quad\text{or}\quad \lambda=\frac{n\pi}{L},\ \ n=1,2,3,\ldots.$$

Here, we exclude $n=0$, since it makes $c=0$. Therefore, the nontrivial solutions (eigenfunctions) X_n corresponding to the eigenvalue $c=-\lambda^2$ are given by

$$X_n(x)=a_n\sin(\frac{n\pi x}{L}), \tag{9}$$

where a_n's are arbitrary constants.

Step 3:(Applying IC)

Let us consider solving equation (4). The general solution to (4) with $c=-\lambda^2=(\frac{n\pi}{L})^2$ is

$$T_n(t)=b_ne^{-\alpha^2(\frac{n\pi}{L})^2t}.$$

Combing this with (9), the product solution $u(x,t) = X(x)T(t)$ becomes

$$\begin{aligned} u_n(x,t) &:= X_n(x)T_n(t) = a_n \sin(\frac{n\pi x}{L}) b_n e^{-\alpha^2(\frac{n\pi}{L})^2 t} \\ &= c_n e^{-\alpha^2(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L}), \quad n = 1, 2, 3, \ldots, \end{aligned}$$

where c_n is an arbitrary constant.

Since the problem (9) is linear and homogeneous, an application of superposition principle gives

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L}),} \tag{10}$$

which will be a solution to (1)-(3), provided the infinite series has the proper convergence behavior.

Since the solution (10) is to satisfy IC (3), we must have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \quad 0 < x < L.$$

Thus, if $f(x)$ has an expansion of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right), \tag{11}$$

which is called a Fourier sine series (FSS) with c_n's are given by the formula

$$c_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx. \tag{12}$$

Then the infinite series (10) with the coefficients c_n given by (12) is a solution to the problem (1)-(3).

Example 1. *Find the solution to the following IBVP:*

$$u_t = 3u_{xx} \quad 0 \le x \le \pi, \ 0 < t < \infty, \tag{13}$$

$$u(0,t) = u(\pi,t) = 0, \quad 0 < t < \infty, \tag{14}$$

$$u(x,0) = 3\sin 2x - 6\sin 5x, \quad 0 \le x \le \pi. \tag{15}$$

Solution. Comparing (13) with (1), we notice that $\alpha^2 = 3$ and $L = \pi$. Using formula (10), we write a solution $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin(nx).$$

To determine c_n's, we use IC (15) to have

$$u(x,0) = 3\sin 2x - 6\sin 5x = \sum_{n=1}^{\infty} c_n \sin(nx).$$

Comparing the coefficients of like terms, we obtain

$$c_2 = 3 \quad \text{and} \quad c_5 = -6,$$

and the remaining c_n's are zero. Hence, the solution to the problem (13)-(15) is

$$\begin{aligned} u(x,t) &= c_2 e^{-3(2)^2 t}\sin(2x) + c_5 e^{-3(5)^2 t}\sin(5x) \\ &= 3e^{-12t}\sin(2x) - 6e^{-75t}\sin(5x). \end{aligned}$$

Practice Problems

1. Solve the following IBVP:

$$\begin{aligned} &u_t = 16u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ &u(0,t) = 0, \; u(1,t) = 0, \quad t > 0, \\ &u(x,0) = (1-x)x, \quad 0 < x < 1. \end{aligned}$$

2. Solve the following IBVP:

$$\begin{aligned} &u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ &u_x(0,t) = u_x(\pi,t) = 0, \quad t > 0, \\ &u(x,0) = 1 - \sin x, \quad 0 < x < \pi. \end{aligned}$$