Lecture 2 The Maximum and Minimum Principle

In this lecture, we shall prove the maximum and minimum properties of the heat equation. These properties can be used to prove uniqueness and continuous dependence on data of the solutions of these equations.

To begin with, we shall first prove the maximum principle for the inhomogeneous heat equation ($F \neq 0$).

Theorem 1. (The maximum principle) *Let $R : 0 \leq x \leq L,\ 0 \leq t \leq T$ be a closed region and let $u(x,t)$ be a solution of*

$$u_t - \alpha^2 u_{xx} = F(x,t) \quad (x,t) \in R, \tag{1}$$

which is continuous in the closed region R. If $F < 0$ in R, then $u(x,t)$ attains its maximum values on $t = 0$, $x = 0$ or $x = L$ and not in the interior of the region or at $t = T$. If $F > 0$ in R, then $u(x,t)$ attains its minimum values on $t = 0$, $x = 0$ or $x = L$ and not in the interior of the region or at $t = T$.

Proof. We shall show that if a maximum or minimum occurs at an interior point $0 < x_0 < l$ and $0 < t_0 \leq T$, then we will arrive at contradiction. Let us consider the following cases.

Case I: First, consider the case with $F < 0$. Since $u(x,t)$ is continuous in a closed and bounded region in R, $u(x,t)$ must attain its maximum in R. Let (x_0, t_0) be the interior maximum point. Then, we must have

$$u_{xx}(x_0,t_0) \leq 0, \qquad u_t(x_0,t_0) \geq 0. \tag{2}$$

Since $u_x(x_0,t_0) = 0 = u_t(x_0,t_0)$, we have

$$u_t(x_0,t_0) = 0 \quad \text{if} \ \ t_0 < T.$$

If $t_0 = T$, the point $(x_0,t_0) = (x_0,T)$ is on the boundary of R, then we claim that

$$u_t(x_0,t_0) \geq 0$$

as u may be increasing at (x_0,t_0). Substituting (2) in (1), we find that the left side of the equation (1) is non-negative while the right side is strictly negative. This leads to a contradiction and hence, the maximum must be assumed on the initial line or on the boundary.

Case II: Consider the case with $F > 0$. Let there be an interior minimum point (x_0, t_0) in R such that

$$u_{xx}(x_0, t_0) \geq 0, \quad u_t(x_0, t_0) \leq 0. \tag{3}$$

Note that the inequalities (3) is same as (2) with the signs reversed. Again arguing as before, this leads to a contradiction, hence the minimum must be assumed on the initial line or on the boundary.

Note: When $F = 0$ i.e., for homogeneous equation, the inequalities (2) at a maximum or (3) at a minimum do not leads to a contradiction when they are inserted into (1) as u_{xx} and u_t may both vanish at (x_0, t_0).

Below, we present a proof of the maximum principle for the homogeneous heat equation.

Theorem 2. (The maximum principle) *Let $u(x, t)$ be a solution of*

$$u_t = \alpha^2 u_{xx} \quad 0 \leq x \leq L, \;\; 0 < t \leq T, \tag{4}$$

which is continuous in the closed region $R : 0 \leq x \leq L$ and $0 \leq t \leq T$. The maximum and minimum values of $u(x, t)$ are assumed on the initial line $t = 0$ or at the points on the boundary $x = 0$ or $x = L$.

Proof. Let us introduce the auxiliary function

$$v(x, t) = u(x, t) + \epsilon x^2, \tag{5}$$

where $\epsilon > 0$ is a constant and u satisfies (4). Note that $v(x, t)$ is continuous in R and hence it has a maximum at some point (x_1, t_1) in the region R.

Assume that (x_1, t_1) is an interior point with $0 < x_1 < L$ and $0 < t_1 \leq T$. Then we find that

$$v_t(x_1, t_1) \geq 0, \quad v_{xx}(x_1, t_1) \leq 0. \tag{6}$$

Since u satisfies (4), we have

$$v_t - \alpha^2 v_{xx} = u_t - \alpha^2 u_{xx} - 2\alpha^2 \epsilon = -2\alpha^2 \epsilon < 0. \tag{7}$$

Substituting (6) into (4) and using (7) now leads to

$$0 \leq v_t - \alpha^2 v_{xx} < 0,$$

which is a contradiction since the left side is non-negative and the right side is strictly negative. Therefore, $v(x, t)$ assumes its maximum on the initial line or on the boundary since v satisfies (1) with $F < 0$.

Let

$$M = \max\{u(x,t)\} \text{ on } t = 0, x = 0, \text{ and } x = L,$$

i.e., M is the maximum value of u on the initial line and boundary lines. Then

$$v(x,t) = u(x,t) + \epsilon x^2 \leq M + \epsilon L^2, \text{ for } 0 \leq x \leq L, \ 0 \leq t \leq T. \quad (8)$$

Since v has its maximum on $t = 0$, $x = 0$, or $x = L$, we obtain

$$u(x,t) = v(x,t) - \epsilon x^2 \leq v(x,t) \leq M + \epsilon L^2. \quad (9)$$

Since ϵ is arbitrary, letting $\epsilon \to 0$, we conclude that

$$u(x,t) \leq M \text{ for all } (x,t) \in R, \quad (10)$$

and this completes the proof.

Remark 3.

- The minimum principle for the heat equation can be obtained by replacing the function $u(x,t)$ by $-u(x,t)$, where $u(x,t)$ is a solution of (4). Clearly, $-u$ is also a solution of (4) and the maximum values of u correspond to the minimum values of u. Since u satisfies the maximum principle, we conclude that u assumes its min-minimum values on the initial line or on the boundary lines. In particular, this implies that if the initial and boundary data for the problem are non- negative, then the solution must be non-negative.

- In geometrical term, the maximum principle states that if a solution of the problem (4) is graphed in the xtu-space, then the surface $u = u(x,t)$ achieves its maximum height above one of the three sides $x = 0, x = L, t = 0$ of the rectangle $0 \leq x \leq L$, $0 \leq t \leq T$.

- From a physical perspective, the maximum principle states that the temperature, at any point x inside the rod at any time t ($0 \leq t \leq T$), is less than the maximum of the initial temperature distribution or the maximum of the temperatures prescribed at the ends during the time interval $[0, T]$.

1 Uniqueness and continuous dependence

As a consequence of the maximum principle, we can show that the heat flow problem has a unique solution and depend continuously on the given initial and boundary data.

Theorem 4. (Uniqueness result) *Let $u_1(x,t)$ and $u_2(x,t)$ be solutions of the following problem*

$$\begin{aligned} \textit{PDE:} \quad & u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \; t > 0, \\ \textit{BC:} \quad & u(0,t) = g(t), \quad u(L,t) = h(t), \\ \textit{IC:} \quad & u(x,0) = f(x), \end{aligned} \tag{11}$$

where $f(x)$, $g(t)$ and $h(t)$ are given functions. Then $u_1(x,t) = u_2(x,t)$, for all $0 \le x \le L$ and $t \ge 0$.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of (11). Set $w(x,t) = u_1(x,t) - u_2(x,t)$. Then w satisfies

$$\begin{aligned} & w_t = \alpha^2 w_{xx} \quad 0 < x < L, \; t > 0, \\ & w(0,t) = 0, \quad w(L,t) = 0, \\ & w(x,0) = 0. \end{aligned}$$

By the maximum principle (cf. Theorem 2), we must have

$$w(x,t) \le 0 \Longrightarrow u_1(x,t) \le u_2(x,t), \; \text{ for all } 0 \le x \le L, \; t \ge 0.$$

A similar argument with $\bar{w} = u_2 - u_1$ yields

$$u_2(x,t) \le u_1(x,t) \; \text{ for all } 0 \le x \le L, \; t \ge 0.$$

Therefore, we have

$$u_1(x,t) = u_2(x,t) \; \text{ for all } 0 \le x \le L, \; t \ge 0,$$

and this completes the proof.

Theorem 5. (Continuous Dependence on the IC and BC) *Let $u_1(x,t)$ and $u_2(x,t)$, respectively, be solutions of the problems*

$$\begin{aligned} u_t = \alpha^2 u_{xx}; \qquad & u_t = \alpha^2 u_{xx} \\ u(0,t) = g_1(t) \;\; u(L,t) = h_1(t); \qquad & u(0,t) = g_2(t) \;\; u(L,t) = h_2(t) \\ u(x,0) = f_1(x); \qquad & u(x,0) = f_2(x), \end{aligned} \tag{12}$$

in the region $0 \le x \le L$, $t \ge 0$. If

$$|f_1(x) - f_2(x)| \le \epsilon \; \textit{ for all } x, \; 0 \le x \le L,$$

and

$$|g_1(t) - g_2(t)| \leq \epsilon \ \ \textit{and} \ \ |h_1(t) - h_2(t)| \leq \epsilon \ \ \textit{for all}\, t, \ 0 \leq t \leq T,$$

for some $\epsilon \geq 0$*, then we have*

$$|u_1(x,t) - u_2(x,t)| \leq \epsilon \ \ \textit{for all}\, x \ \textit{and}\, t, \textit{where} \ \ 0 \leq x \leq L, \ 0 \leq t \leq T.$$

Proof. Let $v(x,t) = u_1(x,t) - u_2(x,t)$. Then $v_t = \alpha^2 v_{xx}$ and we obtain

$$\begin{aligned}
|v(x,0)| &= |f_1(x) - f_2(x)| \leq \epsilon, \ \ 0 \leq x \leq L,\\
|v(0,t)| &= |g_1(t) - g_2(t)| \leq \epsilon, \ \ 0 \leq t \leq T,\\
|v(L,t)| &= |h_1(t) - h_2(t)| \leq \epsilon, \ \ 0 \leq t \leq T.
\end{aligned}$$

Note that the maximum of v on $t = 0$ $(0 \leq x \leq L)$ and $x = 0$ and $x = L$ $(0 \leq t \leq T)$ is not greater than ϵ. The minimum of v on these boundary lines is not less than $-\epsilon$. Hence, the maximum/minimum principle yields

$$-\epsilon \leq v(x,t) \leq \epsilon \implies |u_1(x,t) - u_2(x,t)| = |v(x,t)| \leq \epsilon.$$

Note: (i) We observe that when $\epsilon = 0$, the problems in (12) are identical. We conclude that $|u_1(x,t) - u_2(x,t)| \leq 0$ (i.e. $u_1 = u_2$). This proves the uniqueness result.

(ii) Suppose a certain initial/boundary value problem has a unique solutions. Then a small change in the initial and/or boundary conditions yields a small change in the solutions.

For the inhomogeneous equation (1), we have seen that the maximum or minimum values must be attained either on the initial line or the boundary lines and that they cannot be assumed in the interior. This result is known as a strong maximum or minimum principle.

Theorem 6. (Strong maximum principle) *Let* $u(x,t)$ *be a solution of the heat equation in the rectangle* $R \ : \ 0 \leq x \leq L, \ 0 \leq t \leq T$*. If* $u(x,t)$ *achieves its maximum at* (x^*, T)*, where* $0 < x^* < L$*, then* u *must be constant in* R*.*

Practice Problems

1. Use the maximum/minimum principle to show that the solution u of the problem

$$\begin{aligned}
&u_t = u_{xx}, \ \ 0 < x < \pi, \ \ t > 0,\\
&u_x(0,t) = 0, \ u_x(\pi,t) = 0, \ \ \ t > 0,\\
&u(x,0) = \sin(x) + \frac{1}{2}\sin(2x), \ \ 0 \leq x \leq \pi
\end{aligned}$$

   satisfies $0 \leq u(x,t) \leq \frac{3\sqrt{3}}{4}$, $t \geq 0$.

2. Let $Q = \{(x,t) \,|\, 0 < x < \pi,\ 0 < t \leq T\}$. Let u solves

$$
\begin{aligned}
&u_t = u_{xx} \ \text{ in } \ Q, \\
&u(0,t) = 0,\ u(\pi,t) = 0, \quad 0 \leq t \leq T, \\
&u(x,0) = \sin^2(x), \ \ 0 \leq x \leq \pi.
\end{aligned}
$$

Use maximum principle to show that $0 \leq u(x,t) \leq e^{-t} \sin x$ in Q.