Module 5: Heat Equation

In this module, we shall study one-dimensional heat equation. It is a parabolic partial differential equation which describes diffusion processes such as heat conduction, chemical concentration etc., and hence the heat equation is often called the diffusion equation.

This module consists of five lectures and is organized as follows. The first lecture is devoted to the derivation of heat equation from the principle of conservation of energy. Uniqueness results and the maximum principle for the heat equation will be discussed in the second lecture. Third lecture discusses the method of solution by separation of variables. Fourth and fifth lectures are devoted to the cases where the boundary conditions do not change with time and time-dependent boundary conditions, respectively.

Lecture 1 Modeling the Heat Equation

We shall derive heat equation from the principle of conservation of energy and the fact that heat flows from hot regions to cold regions.

Consider a wire or rod of length L which is made of some heat-conducting material and is insulated on the outside, except possibly over the ends at $x = 0$ and $x = L$. Let $u(x,t)$ denote the temperature at x at time t. $u(x,t)$ is assumed to be constant on each cross section at each time. By the principle of conservation of energy (heat energy), the

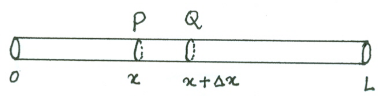

Figure 5.1: A thin rod of length L

net change of heat inside the segment PQ (between x and $x + \Delta x$) is equal to the net heat flux across the boundaries and the total heat generated inside PQ. If c is thermal capacity of the rod, ρ is the density of the rod, A is the cross-section area of the rod, k is thermal conductivity of the rod and $f(x,t)$ is the external heat source, then we calculate these terms as follows:

Total amount of heat inside the segment PQ at time $t = \displaystyle\int_x^{x+\Delta x} c\rho A u(\tau,t)d\tau.$

Net change of heat inside $PQ = \dfrac{d}{dt}\displaystyle\int_x^{x+\Delta x} c\rho A u(\tau,t)ds = c\rho A\int_x^{x+\Delta x} u_t(\tau,t)d\tau.$

Net flux of heat across the boundaries $=kA[u_x(x+\Delta x,t) - u_x(x,t)]$.

Heat generated due to external heat source inside $PQ = A\displaystyle\int_x^{x+\Delta x} f(\tau,t)d\tau.$

By the principle of conservation of energy, we write

$$\frac{d}{dt}\int_x^{x+\Delta x} c\rho A u(\tau,t)d\tau = c\rho A\int_x^{x+\Delta x} u_t(\tau,t)d\tau$$
$$= kA[u_x(x+\Delta x,t) - u_x(x,t)] + A\int_x^{x+\Delta x} f(\tau,t)d\tau. \tag{1}$$

Applying Mean Value Theorem for integral[1], we obtain

$$c\rho A u_t(\xi_1,t)\Delta x = kA[u_x(x+\Delta x,t) - u_x(x,t)] + Af(\xi_2,t)\Delta x,$$

where $\xi_1, \xi_2 \in (x, x+\Delta x)$, and hence,

$$u_t(\xi_1,t) = \frac{k}{c\rho}\left[\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right] + \frac{1}{c\rho} f(\xi_2,t).$$

Now, letting $\Delta x \to 0$, we arrive at

$$\boxed{u_t(x,t) = \alpha^2 u_{xx}(x,t) + F(x,t),} \tag{2}$$

where $\alpha^2 = k/(c\rho)$ is called the thermal diffusivity of the rod and $F(x,t) = \frac{1}{c\rho} f(x,t)$ is called the heat source density.

Remark 1.

- When the rod is not laterally insulated and we allow the heat to flow in and out across the lateral boundary at a rate proportional to the difference between the temperature $u(x,t)$ and the surrounding medium, the conservation of heat principle yields
$$u_t = \alpha^2 u_{xx} - \beta(u - u_0), \quad \beta > 0.$$
The heat loss ($u > u_0$) or gain ($u < u_0$) is proportional to the difference between the temperature $u(x,t)$ of the rod and the surrounding medium u_0. Here, β is the constant of proportionality.

- If the material of the rod is uniform, then k is independent of x. For some materials, the value of k depends on the temperature u and hence the resulting heat equation
$$u_t = \frac{1}{c\rho}\frac{\partial}{\partial x}\left\{k(u)\frac{\partial u}{\partial x}\right\}$$
is nonlinear.

- If the material is nonhomogeneous the diffusion within the rod depends on x. For example, suppose the half of the rod is made of copper and other half is made of steel, then the PDE that describes the heat flow is given by
$$u_t = \alpha^2(x) u_{xx}, \quad 0 < x < L,$$

[1] If $f(x)$ is continuous on $[a,b]$, then there exists at least one number ξ in (a,b) such that
$$\int_a^b f(x)dx = f(\xi)(b-a).$$

with

$$\alpha(x) = \begin{cases} \alpha_1, & 0 < x < L/2, \\ \alpha_2, & L/2 < x < L, \end{cases}$$

where α_1 and α_2 are the thermal diffusivity coefficients of copper and steel, respectively.

Types of BCs: There are three types of boundary conditions that can occur for heat flow problems. They are

- *Dirichlet boundary conditions* (temperature is specified on the boundary):

  Consider heat flow problem in a rod ($0 \leq x \leq L$). The specification of the temperatures $u(0,t)$ and $u(L,t)$ at the ends are classified as Dirichlet type BC.

- *Neumann boundary conditions* (heat flow across the boundary is specified):

  The specification of the normal derivative (i.e., $\frac{\partial u}{\partial n}$, where n is the outward normal to the boundary) on the boundary is classified as Neumann type BCs. For instance, if the end points of a rod is insulated (i.e., we do not allow any flow of heat across the boundary), the BCs are

  $$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad 0 < t < \infty.$$

- *Robin's or Mixed boundary conditions*:

  If the condition on the boundary is a mixture of both Dirichlet and Neumann types i.e.,

  $$\frac{\partial u}{\partial n} = -h(u - g(t))$$

  then it is called Robin's BCs or mixed BCs. Here, h is a constant and $g(t)$ is given function that can vary over the boundary. The mixed BCs may be interpreted as the inward flux across the boundary is proportional to the difference between the temperature u and some specified temperature g. If the temperature u on the boundary is greater than the boundary temperature, then the flow of heat is outward. If u is less than the specified boundary temperature g, then heat flows inward.