3.3 Analysis of Members under Flexure (Part II)

This section covers the following topics.

- Cracking Moment
- Kern Points
- Pressure Line

Introduction

The analysis of flexural members under service loads involves the calculation of the following quantities.

a) Cracking moment.
b) Location of kern points.
c) Location of pressure line.

The following material explains each one of them.

3.2.1 Cracking Moment

The cracking moment (M_{cr}) is defined as the moment due to external loads at which the first crack occurs in a prestressed flexural member. Considering the variability in stress at the occurrence of the first crack, the evaluated cracking moment is an estimate. Nevertheless, the evaluation of cracking moment is important in the analysis of prestressed members.

Based on the allowable tensile stress the prestress members are classified into three types as per **IS:1343 - 1980**. The types are explained in Section 1.2, Advantages and Types of Prestressing. For Type 1 (full prestressing) and Type 2 (limited prestressing) members, cracking is not allowed under service loads. Hence, it is imperative to check that the cracking moment is greater than the moment due to service loads. This is satisfied when the stress at the edge due to service loads is less than the modulus of rupture.

The **modulus of rupture** is the stress at the bottom edge of a simply supported beam corresponding to the cracking moment (M_{cr}). The modulus of rupture is a measure of the flexural tensile strength of concrete. It is measured by testing beams under 2 point loading (also called 4 point loading including the reactions or middle third loading). The modulus of rupture (f_{cr}) is expressed in terms of the characteristic compressive strength (f_{ck}) of concrete by the following equation (**IS:456 - 2000**). Here, f_{cr} and f_{ck} are in N/mm^2.

$$f_{cr} = 0.7\sqrt{f_{ck}} \quad \textbf{(3-3.1)}$$

The following sketch shows the internal forces and the resultant stress profile at the instant of cracking.

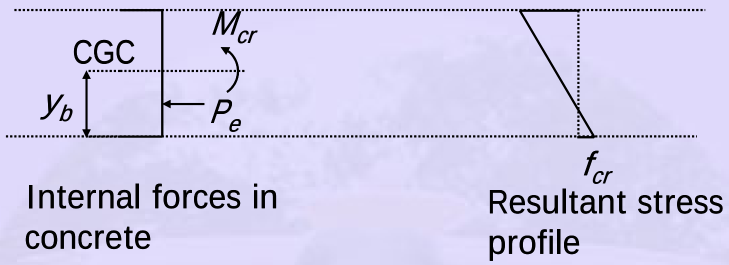


Figure 3-3.1 Internal forces and resultant stress profile at cracking

The stress at the edge can be calculated based on the stress concept as follows. The cracking moment (M_{cr}) can be evaluated by transposing the terms.

$$-\frac{P_e}{A} - \frac{P_e e y_b}{I} + \frac{M_{cr} y_b}{I} = f_{cr}$$

$$\text{or,} \quad \frac{M_{cr} y_b}{I} = f_{cr} + \frac{P_e}{A} + \frac{P_e e y_b}{I}$$

$$\text{or,} \quad M_{cr} = \frac{f_{cr} I}{y_b} + \frac{P_e I}{A y_b} + P_e e \quad \textbf{(3-3.2)}$$

The above equation expresses M_{cr} in terms of the section and material properties and prestressing variables.

3.2.2 Kern Points

When the resultant compression (C) is located within a specific zone of a section of a beam, tensile stresses are not generated. This zone is called the kern zone of a section. For a section symmetric about a vertical axis, the kern zone is within the levels of the upper and lower kern points. When the resultant compression (C) under service loads is located at the upper kern point, the stress at the bottom edge is zero. Similarly, when C at transfer of prestress is located at the bottom kern point, the stress at the upper edge is zero. The levels of the upper and lower kern points from CGC are denoted as k_t and k_b, respectively.

Based on the stress concept, the stress at the bottom edge corresponding to C at the upper kern point, is equated to zero. The following sketch shows the location of C and the resultant stress profile.

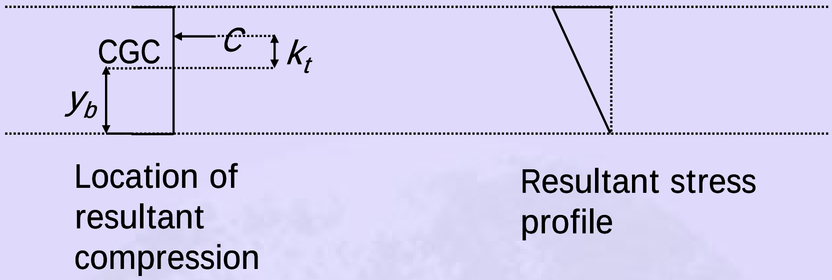


Figure 3-3.2 Resultant stress profile when compression is at upper kern point

The value of k_t can be calculated by equating the stress at the bottom to zero as follows.

$$-\frac{C}{A}+\frac{Ck_ty_b}{I}=0$$

$$\text{or, } -\frac{C}{A}+\frac{Ck_ty_b}{Ar^2}=0$$

$$\text{or, } k_t=\frac{r^2}{y_b} \qquad \textbf{(3-3.3)}$$

The above equation expresses the location of upper kern point in terms of the section properties. Here, r is the radius of gyration and y_b is the distance of the bottom edge from CGC.

Similar to the calculation of k_t, the location of the bottom kern point can be calculated by equating the stress at the top edge to zero. The following sketch shows the location of C and the resultant stress profile.

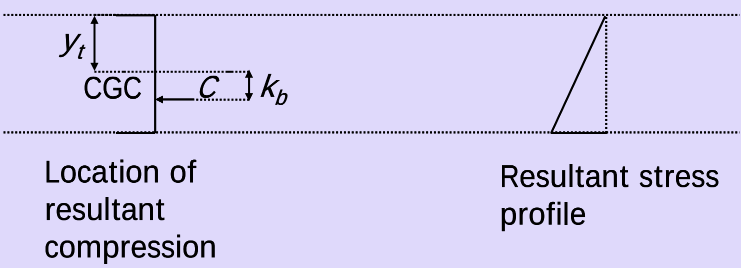


Figure 3-3.3 Resultant stress profile when compression is at lower kern point

$$-\frac{C}{A}+\frac{Ck_b y_t}{I}=0$$

$$\text{or,}\ -\frac{C}{A}+\frac{Ck_b y_t}{Ar^2}=0$$

$$\text{or,}\quad k_b=\frac{r^2}{y_t} \qquad \textbf{(3-3.4)}$$

Here, y_t is the distance of the top edge from CGC.

Cracking Moment using Kern Points

The kern points can be used to determine the cracking moment (M_{cr}). The cracking moment is slightly greater than the moment causing zero stress at the bottom. C is located above k_t to cause a tensile stress f_{cr} at the bottom. The incremental moment is $f_{cr}\, I/y_b$. The following sketch shows the shift in C outside the kern to cause cracking and the corresponding stress profiles.

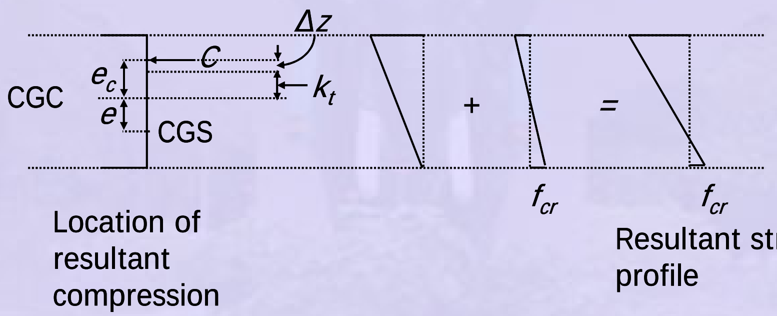


Figure 3-3.4 Resultant stress profile at cracking of the bottom edge

The cracking moment can be expressed as the product of the compression and the lever arm. The lever arm is the sum of the eccentricity of the CGS (e) and the eccentricity of the compression (e_c). The later is the sum of k_t and Δz, the shift of C outside the kern.

$$M_{cr}=C\left(e+e_c\right)$$

$$=C\left(e+k_t+\Delta z\right)$$

$$\text{or,}\ M_{cr}=C\left(e+k_t\right)+\frac{f_{cr}I}{y_b} \qquad \textbf{(3-3.5)}$$

Substituting $C = P_e$, $k_t = r^2/y_b$ and $r^2 = I/A$, the above equation becomes same as the previous expression of M_{cr}.

$$M_{cr} = P_e\left(\frac{r^2}{y_b} + e\right) + \frac{f_{cr}I}{y_b}$$

$$\text{or, } M_{cr} = \frac{f_{cr}I}{y_b} + \frac{P_eI}{Ay_b} + P_ee \qquad \textbf{(3-3.6)}$$

3.2.3 Pressure Line

The pressure line in a beam is the locus of the resultant compression (C) along the length. It is also called the **thrust line** or **C-line**. It is used to check whether C at transfer and under service loads is falling within the kern zone of the section. The eccentricity of the pressure line (e_c) from CGC should be less than k_b or k_t to ensure C in the kern zone.

The pressure line can be located from the lever arm (z) and eccentricity of CGS (e) as follows. The lever arm is the distance by which C shifts away from T due to the moment. Subtracting e from z provides the eccentricity of C (e_c) with respect to CGC. The variation of e_c along length of the beam provides the pressure line.

$$z = \frac{M}{C}$$

$$e_c = z - e \qquad \textbf{(3-3.7)}$$

A positive value of e_c implies that C acts above the CGC and vice-versa. If e_c is negative and the numerical value is greater than k_b (that is $|e_c| > k_b$), C lies below the lower kern point and tension is generated at the top of the member. If $e_c > k_t$, then C lies above the upper kern point and tension is generated at the bottom of the member.

Pressure Line at Transfer

The pressure line is calculated from the moment due to the self weight. The following sketch shows that the pressure line for a simply supported beam gets shifted from the CGS with increasing moment towards the centre of the span.

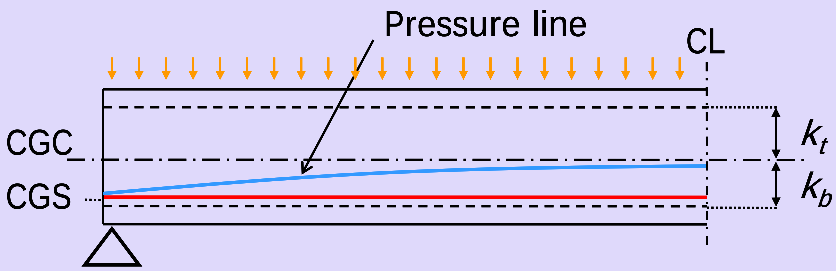


Figure 3-3.4 Pressure line at transfer

Pressure Line under Service Loads

The pressure line is calculated from the moment due to the service loads. The following sketch shows that the pressure line for a simply supported beam gets further shifted from the CGS at the centre of the span with increased moment under service condition.

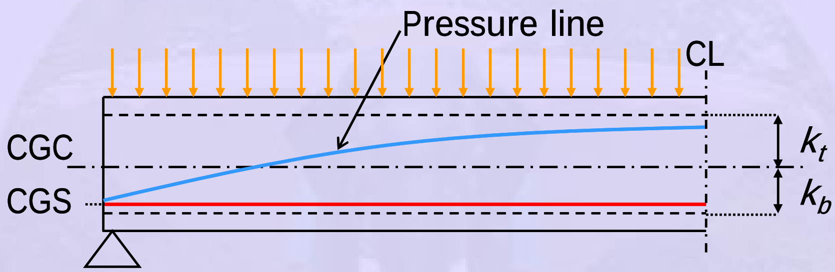


Figure 3-3.4 Pressure line under service loads

Limiting Zone

For fully prestressed members (Type 1), tension is not allowed under service conditions. If tension is also not allowed at transfer, *C* always lies within the kern zone. The limiting zone is defined as the zone for placing the CGS of the tendons such that *C* always lies within the kern zone.

For limited prestressed members (Type 2 and Type 3), tension is allowed at transfer and under service conditions. The limiting zone is defined as the zone for placing the CGS such that the tensile stresses in the extreme edges are within the allowable values.

The following figure shows the limiting zone (as the shaded region) for a simply supported beam subjected to uniformly distributed load.

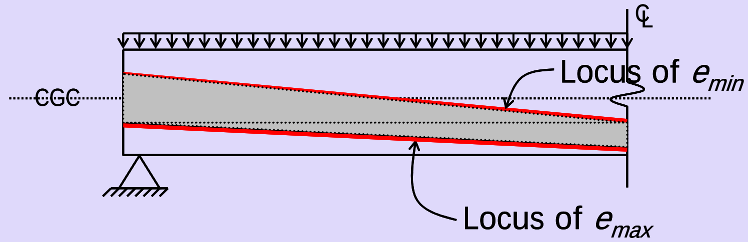
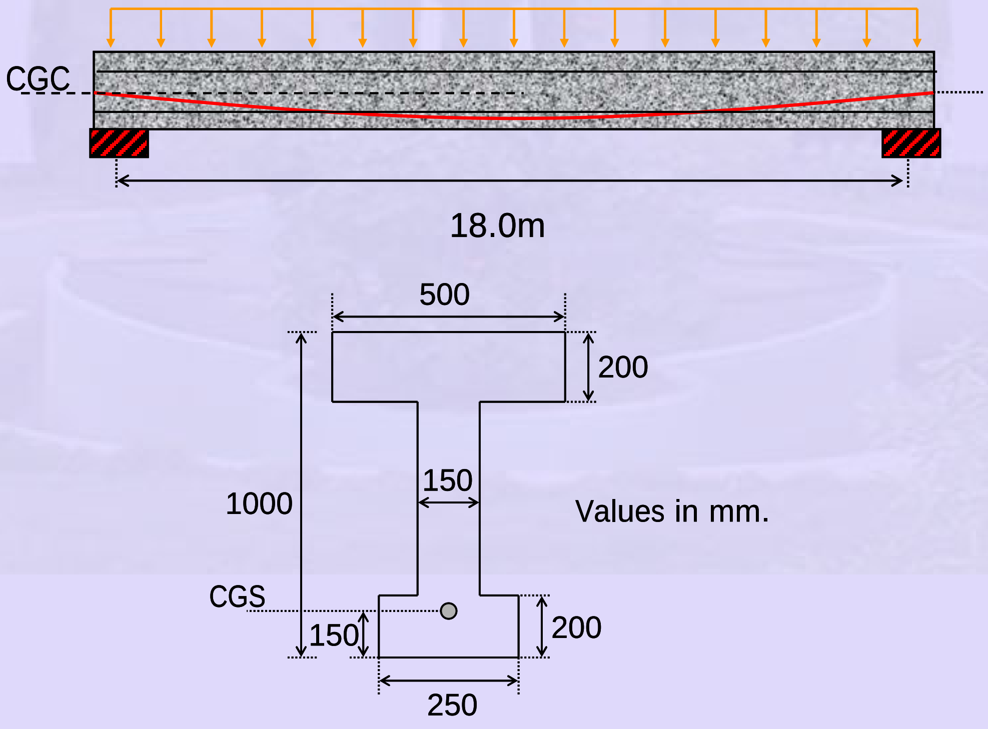


Figure 3-3.4 Limiting zone for a simply supported beam

The determination of limiting zone is given in Section 4.4, Design of Sections for Flexure (Part III).

Example 3-3.1

For the post-tensioned beam with a flanged section as shown, the profile of the CGS is parabolic, with no eccentricity at the ends. The live load moment due to service loads at mid-span (M_{LL}) is 648 kNm. The prestress after transfer (P_0) is 1600 kN. Assume 15% loss at service. Grade of concrete is M30.



Cross-section at mid-span

Evaluate the following quantities.

a) **Kern levels**

b) **Cracking moment**

c) **Location of pressure line at mid-span at transfer and at service.**

d) **The stresses at the top and bottom fibres at transfer and at service.**

Compare the stresses with the following allowable stresses at transfer and at service.

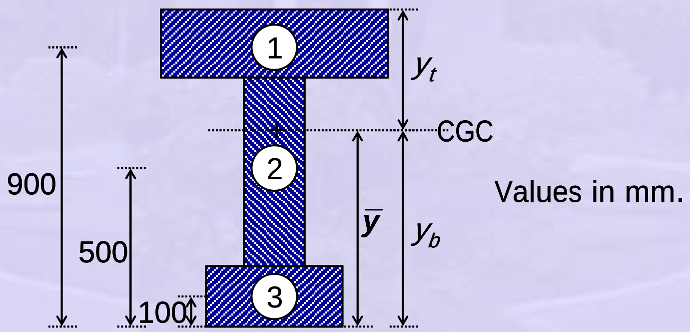
For compression, $f_{cc,all}$ **= – 18.0 N/mm^2**

For tension, $f_{ct,all}$ **= 1.5 N/mm^2.**

Solution

Calculation of geometric properties

The section is divided into three rectangles for the computation of the geometric properties. The centroid of each rectangle is located from the soffit.



Area of the section

Area of 1 $= A_1 = 500 \times 200$

$= 100{,}000 \text{ mm}^2$

Area of 2 $= A_2 = 600 \times 150$

$= 90{,}000 \text{ mm}^2$

Area of 3 $= A_3 = 250 \times 200$

$= 50{,}000 \text{ mm}^2$

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= 240{,}000 \text{ mm}^2 \end{aligned}$$

Location of CGC from the soffit

$$\begin{aligned} \bar{y} &= \frac{A_1 \times 900 + A_2 \times 500 + A_3 \times 100}{A} \\ &= 583.3 \text{ mm} \end{aligned}$$

Therefore,

$$\begin{aligned} y_b &= 583.3 \text{ mm} \\ y_t &= 1000.0 - 583.3 \\ &= 416.7 \text{ mm} \end{aligned}$$

Eccentricity of CGS at mid-span

$$\begin{aligned} e &= \bar{y} - 150 \\ &= 583.3 - 150 \\ &= 433.3 \text{ mm} \end{aligned}$$

Moment of inertia of 1 about axis through CGC

$$\begin{aligned} I_1 &= \frac{1}{12} \times 500 \times 200^3 + A_1 \times (900 - 583.3)^2 \\ &= 1.036 \times 10^{10} \text{ mm}^4 \end{aligned}$$

Moment of inertia of 2 about axis through CGC

$$\begin{aligned} I_2 &= \frac{1}{12} \times 150 \times 600^3 + A_2 \times (583.3 - 500)^2 \\ &= 3.32 \times 10^9 \text{ mm}^4 \end{aligned}$$

Moment of inertia of 3 about axis through CGC

$$\begin{aligned} I_3 &= \frac{1}{12} \times 250 \times 200^3 + A_3 \times (583.3 - 100)^2 \\ &= 1.184 \times 10^{10} \text{ mm}^4 \end{aligned}$$

Moment of inertia of the section

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= (1.036 + 0.336 + 1.184) \times 10^{10} \\ &= 2.552 \times 10^{10} \text{ mm}^4 \end{aligned}$$

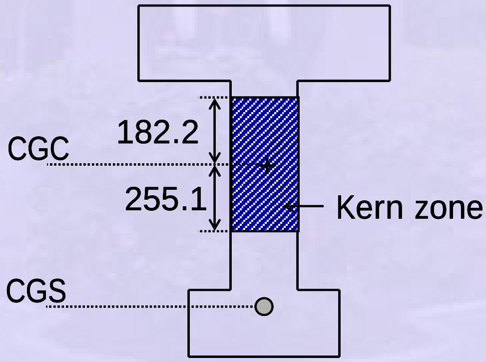
Square of the radius of gyration

$$\begin{aligned} r^2 &= \frac{I}{A} \\ &= \frac{2.552 \times 10^{10}}{240,000} \\ &= 1.063 \times 10^5 \text{ mm}^2 \end{aligned}$$

a) Kern levels of the section

$$\begin{aligned} k_t &= \frac{r^2}{y_b} \\ &= \frac{1.063 \times 10^5}{583.3} \\ &= 182.2 \text{mm} \end{aligned}$$

$$\begin{aligned} k_b &= \frac{r^2}{y_t} \\ &= \frac{1.063 \times 10^5}{416.7} \\ &= 255.1 \text{mm} \end{aligned}$$



Values in mm.

Calculation of moment due to self weight (M_{DL}).

$$\begin{aligned} w_{DL} &= 24.0 \text{ kN/m}^3 \times 240,000 \text{ mm}^2 \times \left(\frac{1}{10^3}\right)^2 \frac{\text{m}^2}{\text{mm}^2} \\ &= 5.76 \text{ kN/m} \end{aligned}$$

$$\begin{aligned} M_{DL} &= \frac{w_{DL} L^2}{8} \\ &= \frac{5.76 \times 18.0^2}{8} \\ &= 233.3 \text{ kNm} \end{aligned}$$

b) Calculation of cracking moment

Modulus of rupture
$$\begin{aligned} f_{cr} &= 0.7\sqrt{f_{ck}} \\ &= 0.7\sqrt{30} \\ &= 3.83\,\text{kN/mm}^2 \end{aligned}$$

$$\begin{aligned} M_{cr} &= \frac{f_{cr} I}{y_b} + \frac{P_e I}{A y_b} + P_e e \\ &= \frac{3.83 \times 2.552 \times 10^{10}}{583.3} + \frac{0.85 \times 1600 \times 10^3 \times 2.552 \times 10^{10}}{240 \times 10^3 \times 583.3} \\ &\quad + 0.8 \times 1600 \times 10^3 \times 433.3\,\text{Nmm} \\ &= 167.6 + 247.9 + 554.6 \\ &= 970.1\,\text{kNm} \end{aligned}$$

Live load moment corresponding to cracking

$$\begin{aligned} M_{LL\ cr} &= 970.1 - 233.3 \\ &= 736.8\,\text{kNm} \end{aligned}$$

Since the given live load moment (648.0 kNm) is less than the above value, the section is uncracked.

$\Rightarrow$ The moment of inertia of the gross section can be used for computation of stresses.

c) Calculation of location of pressure line at mid-span

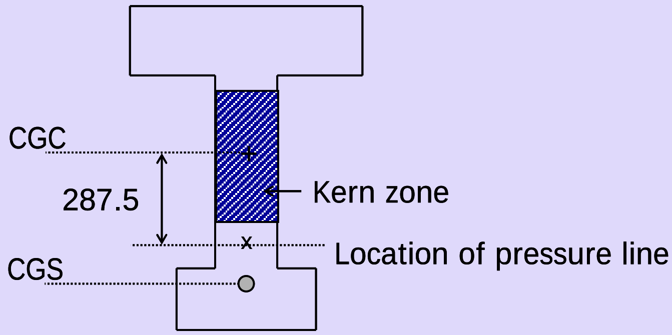
At transfer

$$\begin{aligned} z &= \frac{M_{DL}}{C} \\ &= \frac{233.3 \times 10^3}{1600} \\ &= 145.8\,\text{mm} \end{aligned}$$

$$\begin{aligned} e_c &= z - e \\ &= 145.8 - 433.3 \\ &= -287.5\,\text{mm} \end{aligned}$$

Since e_c is negative, the pressure line is below CGC.

Since the magnitude of e_c is greater than k_b, there is tension at the top.



Value in mm.

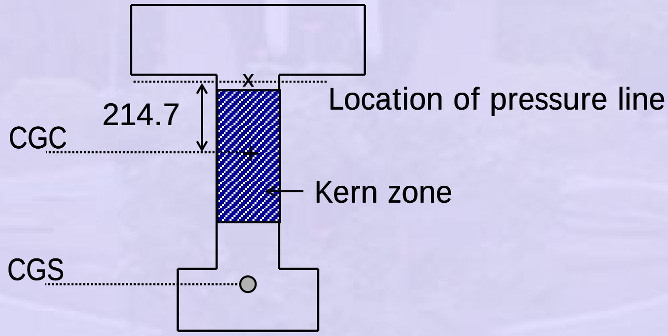
At service

$$z = \frac{M_{DL+LL}}{C}$$
$$= \frac{(233.3 + 648.0)\times 10^3}{0.85\times 1600}$$
$$= 648.0\,\text{mm}$$

$$e_c = z - e$$
$$= 648.0 - 433.3$$
$$= 214.7\,\text{mm}$$

Since e_c is positive, the pressure line is above CGC.

Since the magnitude of e_c is greater than k_t, there is tension at the bottom.

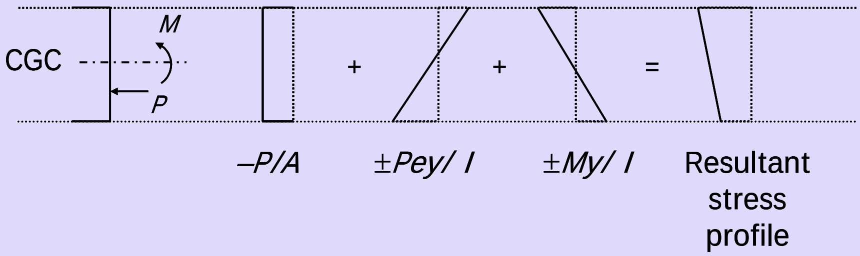


Value in mm.

d) Calculation of stresses

The stress is given as follows.

$$f = -\frac{P}{A} \pm \frac{Pey}{I} \pm \frac{My}{I}$$



Calculation of stresses at transfer ($P = P_0$)

$$\frac{P_0}{A} = -\frac{1600\times10^3}{240\times10^3}$$
$$= -6.67\ \text{N/mm}^2$$

Stress at the top fibre

$$\frac{P_0 e y_t}{I} = \frac{1600\times10^3\times433.3\times416.7}{2.552\times10^{10}}$$
$$= 11.32\ \text{N/mm}^2$$

$$\frac{M_{DL} y_t}{I} = -\frac{233.3\times10^6\times416.7}{2.552\times10^{10}}$$
$$= -3.81\ \text{N/mm}^2$$

$$\therefore f_{c_t} = -6.67 + 11.32 - 3.81$$
$$= 0.84\ \text{N/mm}^2$$

Stress at the bottom fibre

$$\frac{P_0 e y_b}{I} = -\frac{1600\times10^3\times433.3\times583.3}{2.552\times10^{10}}$$
$$= -15.85\ \text{N/mm}^2$$

$$\frac{M_{DL} y_b}{I} = \frac{233.3\times10^6\times583.3}{2.552\times10^{10}}$$
$$= 5.33\ \text{N/mm}^2$$

$$\therefore f_{c_b} = -6.67 - 15.85 + 5.33$$
$$= -17.19\ \text{N/mm}^2$$

Calculation of stresses at service ($P = P_e$)

$$\frac{P_e}{A} = 0.85\frac{P_0}{A}$$
$$= -5.67\ \text{N/mm}^2$$

Stress at the top fibre

$$\frac{P_f e y_t}{I} = 0.85\times11.32$$
$$= 9.62$$

$$\frac{M_{LL} y_t}{I} = -\frac{648.0\times10^6\times416.7}{2.552\times10^{10}}$$
$$= -10.58\ \text{N/mm}^2$$

$$\therefore f_{c_t} = -5.67 + 9.62 - 3.81 - 10.58$$
$$= -10.44 \text{ N/mm}^2$$

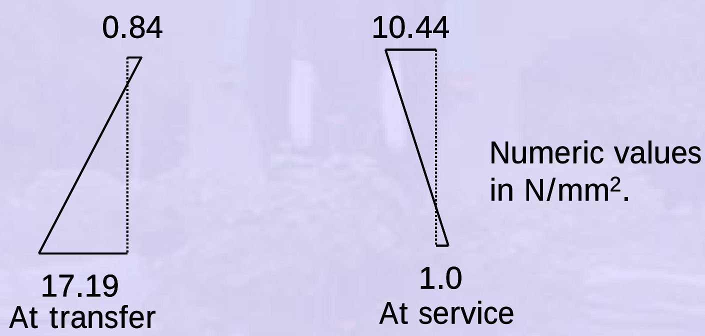
Stress at the bottom fibre

$$\frac{P_f e y_b}{I} = -0.85 \times 15.85$$
$$= -13.47 \text{ N/mm}^2$$

$$\frac{M_{LL} y_b}{I} = \frac{648.0 \times 10^6 \times 583.3}{2.552 \times 10^{10}}$$
$$= 14.81 \text{ N/mm}^2$$

$$\therefore f_{c_b} = -5.67 - 13.47 + 5.33 + 14.81$$
$$= 1.0 \text{ N/mm}^2$$

The stress profiles are shown.



The allowable stresses are as follows.

For compression, $f_{c,comp}$ = – 18.0 N/mm²

For tension, $f_{c,tens}$ = 1.5 N/mm².

Thus, the stresses are within the allowable limits.