Teacher’s Enrichment Workshop (TEW) on Algebra and Multivariable Calculus

Lectures:

I. Lectures by Dr. D. K. Khurana (May 28-30, 2018)

In his six lectures of one hour each from May 28-30, 2018, the following topics were covered.

Vector spaces and their basic properties, Existence of basis and invariance of its cardinality, Linear Transformations and Matrices, Change of Basis, Triangulation and Diagonalization, Modules, Modules versus Vector Spaces, Free Modules, Matrices over PIDs, Smith Normal Form, Finitely Generated Modules over PIDs and their structure, Rational and Jordan Canonical Forms.

In the discussion sessions, the following exercises were discussed.

1) Suppose V is an n −dimensional vector space over a field of order m. Find the number of basis of V.
2) Let V and F be as above and L be a linearly independent set with k elements. Find the number of ways in which L can be extended to a basis of V.
3) Find the order of $GL_n(F)$ and $SL_n(F)$, where F is a field of order m.
4) Let W_1 and W_2 be two isomorphic subspaces of a finite dimensional vector space V. Prove that there exists a subspace U such that $V = W_1 \oplus U = W_2 \oplus U$. Show that this is not true if V is infinite dimensional.
5) Let W_1, W_2 and W_3 be three isomorphic subspaces of a finite dimensional vector space V. Does there exist a subspace U such that $V = W_1 \oplus U = W_2 \oplus U = W_3 \oplus U$?
6) Let f and g be two linear transformations of a finite dimensional vector space V such that $fg = I$. Prove that $gf = I$. Show that this is not true if V has infinite dimension.
7) Suppose V is a 2 −dimensional vector space over a field of order m. Find the number of idempotents linear transformations from $V \to V$.
8) Let $A \in \mathbb{M}_n(F)$, where F is a field, be a nilpotent matrix. Prove that the trace of A is zero.
9) Find the number of nilpotent linear transformations in $\mathbb{M}_n(F)$, where F is a finite field.
10) Prove that any characteristic root of a square matrix over a field is also a root of its minimal polynomial.
11) Let V_i be finite dimensional vector spaces with basis $B_i, i = 1,2,3$ and $f: V_2 \to V_3$ and $g: V_1 \to V_2$ be linear transformations. Prove that $[fg]_{B_1B_3} = [g]_{B_1B_2}[f]_{B_2B_3}$, where $[f]_{B_2B_3}$ denotes the matrix of f with respect to bases B_2, B_3 and so on.
12) Let $f: V \to V$ be a linear transformation of a finite dimensions vector space V. Prove that there exists an invertible linear transformation $g: V \to V$ such that $f = fgf$. Show that the result does not hold if dimension of V is infinite.
13) Let $f: V \to V$ be a linear transformation of a finite dimensions vector space V_F with basis B, $A = [f]_B$ and $R = F[x]$. Then prove that
    - (i) As R −modules $V \cong R^n/(A - xI)R^n$.
    - (ii) If $diag(f_1, f_2, \ldots, f_k)$ is the Smith normal form of $A - xI$, then prove that $diag(C(f_1), C(f_2), \ldots, C(f_k))$ is the rational canonical form of A where $C(f)$ denotes the companion matrix of f.
14) Find the rational canonical form of $A = \begin{pmatrix} 0 & -1 & 2 \\ 3 & -4 & 6 \\ 2 & -2 & 3 \end{pmatrix}$ over $\mathbb{Q}$.

II. Lectures by Dr. Rahul Kitture (May 28-30, 2018)

Day 1 (May 28, 2018)

In first lecture, he revised *Groups* through *symmetries,* which provided a quick link to *Group Action* and discussed several examples of finite groups by considering them as symmetries of some objects,

Speakers:

1. Dr Dinesh Khurana, Panjab University, Chandigarh. **(DK)**
2. Dr Rahul Kitture, IISER Mohali. **(RK)**
3. Dr Surya Ramana, HRI, Allahabad. **(SR)**
4. Dr Chanchal Kumar, IISER Mohali. **(CK)**

Time Table followed:

I. Schedule for Day-1 (May 28, 2018)

9:00-9:30 Registration
9:30-10:30 Inauguration by Vice Chancellor and Dean Academics
10:30-11:00 Tea and Snacks

Date	**11:00-12:00**	**12:00-1:00**	**1:00-2:00**	**2:00-3:00**	**3:00-4:00**	**4:00-4:30**	**4:30-5:30**
May 28	DK	RK	Lunch	DK	RK	Tea	DK, RK

II. Schedule for rest of the Days

Date	**9:30-10:30**	**10:30-11:00**	**11:00-12:00**	**12:00-1:00**	**1:00-2:30**	**2:30-3:30**	**3:30-4:00**	**4:00-5:00**
May 29	DK	Tea and Snacks	DK	RK	Lunch	RK	Tea	RK
May 30	DK	Tea and Snacks	RK	DK	Lunch	RK	Tea	DK
May 31	SR	Tea and Snacks	SR	CK	Lunch	CK	Tea	SR, CK
June 01	SR	Tea and Snacks	SR	CK	Lunch	CK	Tea	CK
June 02	SR	Tea and Snacks	SR	CK	Lunch	CK	Tea	SR

such as Klein-4 group, Cyclic groups, Dihedral Groups, and A_4. Dr. Rahul introduced basic notions in Group action and proved orbit-stabilizer theorem. As an interesting example, through group of symmetries of a square, which was familiar to everyone, it was easy to verify the three theorems of Sylow explicitly (for Sylow-*2* subgroups). In the second lecture, he discussed simple examples of group action through symmetries of some objects, in which one can naturally see the orbits, stabilizers, and verify the orbit-stabilizer theorem. Specifically he discussed the orders of groups of symmetries of platonic solids.

Day 2 (May 29, 2018)

In first lecture, the topic was the action of a group on a set, which provided a homommorphism from the group to the permutation group of the set under consideration. As a simple application, he illustrated the structure of groups of symmetries of platonic solids. In the second lecture, he discussed the groups of isometries of $\mathbb{R}^n$ with main aim towards the classification of finite subgroups of it. With this focus, he proved some important theorems about structure of an isometry (composition of translation and orthogonal transformation). In the tutorial session, he discussed following problems

1) In any group action, any two orbits are either equal or disjoint.
2) Elements of same orbit have conjugate stabilizers, and in particular, have same order.
3) Discuss the group of rotational symmetries of tetrahedron.
4) The general form of 2×2 orthogonal matrix is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Show that the first matrix represents rotation and second matrix represents reflection in a line through origin with angle $\frac{\theta}{2}$ with x −axis.
5) Any orthogonal matrix has determinant ± 1.
6) If A is a 3×3 orthogonal matrix with determinant 1 then A has an eigenvector with eigenvalue 1.

Day 3 (May 30, 2018)

In first lecture, he discussed Burnside's theorem on counting number of orbits for action of finite group on a finite set and the finite subgroups of 2×2 orthogonal matrices. As a primary step in classification of finite subgroups of $\mathbb{O}_3(\mathbb{R})$, he introduced poles on sphere for a rotation, and that a finite subgroup of $\mathbb{O}_3(\mathbb{R})$, permutes (acts on) poles of its (non-identity) elements. This allowed them to locate possible vertices of a regular body inside unit sphere on which G is acting. The explicit discussion (proof) of finite subgroups of $\mathbb{O}_3(\mathbb{R})$, involves some amount of pure algebraic computations and some amount of geometry. In the next lecture, he stressed on geometric part of the proof with attention towards how the platonic solids arise for finite subgroup of $\mathbb{O}_3(\mathbb{R})$ under consideration. Instead of highlighting the five possible groups, he discussed one example in detail namely how the octahedron arises, as a case, in the classification theorem, explicitly.

Lectures by Dr. D. Surya Ramana (May 31 to June 02, 2018)

In his six lectures of one hour each and one and a half tutorial sessions from May 31 to June 02, 2018, the following topics in Multivariable Calculus were covered.

1) The Euclidean structure of $\mathbb{R}^n$; Cauchy-Schwarz inequality.
2) Lipshitz functions, contraction mappings.
3) Fixed point theorem, application to calculation of square roots.s
4) $\mathbb{R}$ Linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, norms on the space of such mappings.
5) Differentiability in several real variables; Derivatives and basic examples.
6) Hadamard's lemma and the chain rule.
7) Directional derivatives, partial derivative ad the Jacobian matrix.

8) The mean value theorem.
9) The implicit function theorem.

IV. Lectures by Dr. Chanchal Kumar (May 31 to June 02, 2018)

Dr. Chanchal gave a series of six lectures on *Elementary Group Theory* and conducted one and a half tutorial sessions in the workshop. The basic notions of Groups, Sub-groups, Quotient groups, Group homomorphisms, and Isomorphism theorems were introduced in the first two lectures on May 31, 2018. Actions of groups on sets were discussed with many examples on June 01, 2018. Using group actions, Cauchy's and Sylow's theorems were proved in the lectures. Applications of Sylow's theorems were also discussed. On June 02, 2018, direct product and semi-direct products of groups were introduced and classification of all groups of order ≤ 16 were obtained. A tutorial question set having more than 20 problems were given to the participants on the first day and these problems were discussed in the tutorial as well as in lecture sessions.

Feedback:

The feedbacks for all the speakers and also the feedbacks for the entire workshop were collected on the prescribed forms from all the participants. We are happy with the feedback as the same is very positive. All the dully-filled feedback forms have been sent to NCM by post. On the last day certificates were distributed among the participants who had attended the entire workshop.

Dr D.K. Khurana (Academic Convener)	Dr. Ajay Kumar (Convener)	Dr. Suraj Goyal (Convener)